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On the Existence of Best Analytic Approximations

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1. INTRODUCTION

A large class of harmonic functions on the closed unit disk have unique best uniform analytic approximations to them [3]. In this paper we show that best analytic approximations to harmonic functions do not always exist by constructing a counterexample from a familiar one concerning the conjugate function problem. Our method is based on what is now a standard approach to extremal problems in H^p spaces.

2. THE H^p CASE

The following pair of theorems have proved to be fundamental to many extremal problems, e.g., in [1] and [2]. Let X be a normed linear space with subspace M. Let $M^\perp$ denote the annihilator of M in X^* the dual of X.

THEOREM 1. *If $x_0 \in X$ then*

$$\inf_{x \in M} \| x_0 - x \| = \max_{L \in M^\perp; \|L\| \leqslant 1} | L(x_0) |.$$

THEOREM 2. *If $L_0 \in X^*$ then*

$$\min_{L \in M^\perp} \| L_0 - L \| = \sup_{x \in M; \|x\| \leqslant 1} | L_0(x) |.$$

For g in $L^p(T)$, $1 < p < \infty$ and $T =$ the unit circle, the problem $\inf_{f \in H^p} \| g - f \|_p$ has a unique minimal solution f in H^p. The existence follows from Theorem 2 with $X^* = L^p$ and $M^\perp = H^p$ while the uniqueness follows from Theorem 1 and the representation of L in $M^\perp$ where $X = L^p$ and $M = H^p$ [2, Theorem 8.1, p. 132].

Let D be the closed unit disk, $T = \partial D$, and A be the set of functions analytic in D and continuous on T so that A is a subspace of $C(T)$, the continuous functions on T. Since $A^{\perp} = H_0^1$, a subspace of $L^1(T)$ and $(H_0^1)^{\perp} = H^{\infty}$, a subspace of $L^{\infty}(T)$, Theorems 1 and 2 imply the following. (See [2], p. 132.)

THEOREM 3. *For every g in $L^{\infty}(T)$ a best approximation f from H^{∞} exists, and if g is continuous f is unique.*

For many g in $C(T)$ the unique best H^{∞} approximation is also continuous, i.e., in A. (See [3].) To show that this is not always true we will need the following comparison between uniform approximation in $C(T)$ and L^{∞} approximation in $L^{\infty}(T)$.

THEOREM 4. *If $g \in C(T)$ then*

$$\inf_{f \in A} \| g - f \| = \inf_{f \in H^{\infty}} \| g - f \|_{\infty} .$$

In other words, the distance from g to A in the uniform norm is equal to the distance of g to H^{∞} in L^{∞}. Since $A \subset H^{\infty}$ the left side is larger than the right side. However, since $A^{\perp} = H_0^1 \subset (H^{\infty})^{\perp}$ in $(L^{\infty})^*$, Theorem 1 shows that the left side is less than the right side. Hence equality must hold.

3. THE CONTINUOUS CASE

THEOREM 5. *There exist harmonic functions for which best analytic approximations in the uniform norm do not exist.*

Consider the function $F(z) = \sum_2^{\infty} (-iz^n/n \log n)$. On T,

$$F(e^{i\theta}) = u(\theta) + i\,v(\theta) = \sum_2^{\infty} \frac{\sin n\theta}{n \log n} + i \sum_2^{\infty} \frac{-\cos n\theta}{n \log n} .$$

The behavior of $u(\theta)$ and $v(\theta)$ is well known [4]. $u(\theta)$ is continuous on T and $v(\theta)$ is continuous on $T - \{1\}$, but as θ approaches 0, $|v(\theta)|$ approaches ∞. In other words, $\mathrm{Re}(F(z)) \in C(T)$ but $F(z)$ is not in A. Let

$$g(e^{i\theta}) = e^{-i\theta}e^{-iv} - e^{-(u+iv)} = e^{-iv}(e^{-i\theta} - e^{-u}) \tag{1}$$

for $\theta \neq 0$ and let $g(1) = 0$. Then $g(e^{i\theta})$ is continuous on T since e^{-iv} is bounded for $\theta \neq 0$ and $(e^{-i\theta} - e^{-u})$ is continuous on T and approaches 0 as θ approaches 0. We will show that g does not have a best uniform approximation from A.

Let g be extended harmonically to all of D. Equation (1) shows that $\inf_{f \in H^{\infty}} \| g - f \|_{\infty} \leqslant 1$ since $e^{-F} = e^{-(u+iv)}$ is in H^{∞}.

On the other hand, the measure $e^{i\theta}e^{u+iv}(d\theta/2\pi)$ on T annihilates H^∞. Let $\alpha = (1/2\pi)\int_0^{2\pi} e^u\, d\theta$. Then $L(\ ) = (1/2\pi\alpha)\int_0^{2\pi} (\ )\, e^{i\theta}e^{u+iv}\, d\theta$ annihilates H^∞, has norm 1, and $L(g) = 1$. Therefore $\inf_{f\in H^\infty} \| g - f \|_\infty \geq 1$ by Theorem 1. Consequently $e^{-F} = e^{-(u+iv)}$ is the unique best H^∞ approximation to g by Theorem 3. Hence by Theorem 4 no best uniform approximation to g from A exists since $e^{-(u+iv)}$ is not in A.

References

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